Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
Prof. Dr. Stefan Helber

Seat number:

Summer semester 2023
Exam for the lecture
"Operations Research"

Please note:

- The exam consists of **12** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice. You may also use an English language dictionary.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	5	Sum
Points						

1. **Improvement search: (10 P.)**

   (a) Assume we are minimizing the function $f(w_1, w_2) = 3w_1^3 w_2^2$.

      i. Construct at point $\mathbf{w} = (w_1, w_2) = (2, 2)$ an improving direction from the gradient of this function. (2 P.)

      ii. Determine by an appropriate gradient test whether at point $\mathbf{w} = (w_1, w_2) = (1, 2)$ the direction $\Delta\mathbf{w} = (-2, 1)$ improves on the objective function. (2 P.)

(b) Explain briefly what characterizes in an improving search an "active constraint" and what a "feasible direction". (2 P.)

(c) Which fundamental properties must the objective function and the feasible set of a problem possess such that the improving search algorithm 3A can be guaranteed to find an optimal solution if one exists? Why? (4 P.)

2. **Branch & Bound** **(15 P.)**

Consider the following integer linear program with a **minimization** objective function:

$$
\begin{aligned}
&\text{Min } 3x_1 + 2x_2 \\
&\text{s.t.} \\
&12x_1 + 2x_2 \geq 23 \\
&2x_1 + 10x_2 \geq 27 \\
&x_i \in \{0, 1, 2, 3, 4, ...\}, \qquad\qquad i \in \{1, 2\}
\end{aligned}
$$

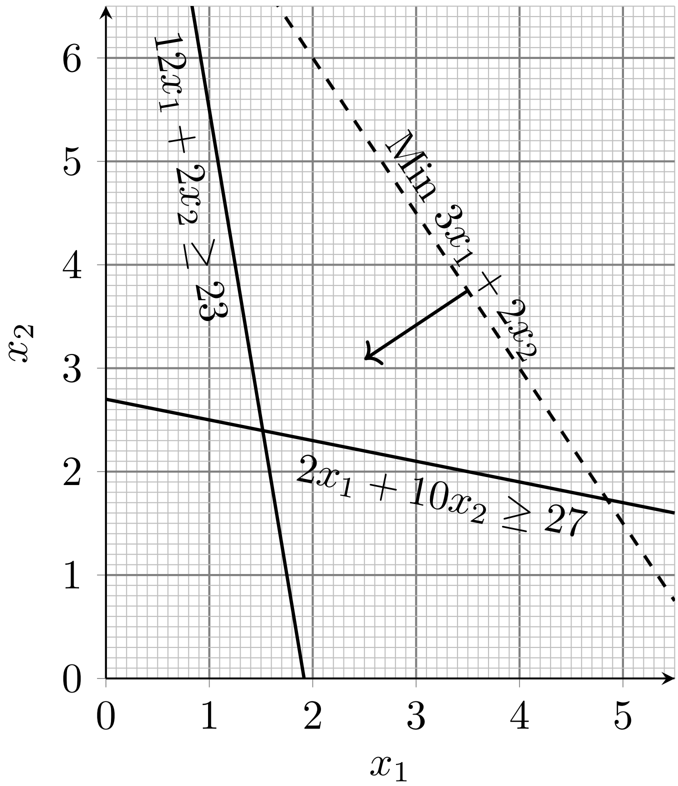
Determine the optimal solution of this integer **minimization** problem by applying a branch&bound algorithm according to the the following specifications:

- Perform a depth-first search!
- Start with incumbent solution ($x_1 = 5,\ x_2 = 5$).
- If in the linear programming relaxation of a candidate problem both variables x_1 and x_2 should be fractional, break ties in favor of the first variable, i.e., branch on x_1.
- When branching on a fractional variable x_i by creating two new problems, create the new problem to be examined next in the depth-first search by rounding up and the other new problem (to be examined later in the depth-first search) by rounding down.
- Number the problems according to the sequence in which you determine their relaxations and make branching or bounding decisions in the decision tree based on the solution of the candidate problem.

Hints and tasks:

- You can determine the values of relaxed variables x_1 and x_2 in a relaxation of a candidate problem to a sufficient degree of accuracy by modifying and reading from the figure on the following page. On this basis you can compute to a sufficient degree of accuracy the objective function values as well.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.

The following figure (see next page) shows the visualization of the objective function and the constraints:



$12x_1 + 2x_2 \geq 23$
Min $3x_1 + 2x_2$
$2x_1 + 10x_2 \geq 27$
x_2
x_1
0
1
2
3
4
5
6

3. **Linear Programming** **(14 P.)**

Consider the following linear program (LP), denoted as the “primal” problem:

$$
\begin{aligned}
\text{Min } & 11x_1 + 10x_2 \\
\text{subject to} & \\
& x_1 \geq 3 \\
& x_2 \geq 4 \\
& x_1 + x_2 \geq 6 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

(a) Determine by inspection

i. the optimal solution to the problem, (2 P.)

ii. the objective function value for the optimal solution, (1 P.)

iii. the values of the dual variables in the optimal solution. (2 P.)

(b) Using your results, show that for all three constraints the primal complementary slackness condition holds! (3. P)

(c) Determine the dual program corresponding to the primal LP! (2 P.)

(d) Give the standard form of the primal LP! (2 P.)

(e) Does the standard form version of the primal LP possess variables which can never be non-basic? If so, give those variable(s). In either case, justify your answer! (2 P.)

4. **Column generation** **(3 P.)**

Assume you are using a column generation algorithm to solve a linear program with a **maximization** (!!) objective function. When does the process or generating new columns or solution patterns stop? Explain!

5. **Models and Modeling:** **(18 P.)**

(a) Write down algebraically an example of a linear programming model with two decision variables x_1 and x_2 which is feasible and possesses infinitely many optimal solutions with respect to those variables x_1 and x_2. Briefly explain which of the solutions can be found by a simplex algorithm! (4 P.)

(b) Draw graphically an example of a linear program with two decision variables x_1 and x_2 in which multiple different (!) basic solutions of the model in standard form correspond to the same optimal solution vector $x = (x_1^*, x_2^*)$. (4 P.)

(c) Write down an example of a small or even tiny feasible non-linear program and give a hint as to why it is non-linear! (4 P.)

(d) Consider the following linear program:

$$\begin{aligned}
&\text{Maximize } 3x_1 + 4x_2 \\
&\text{subject to} \\
&7x_1 + 3x_2 \leq 21 \\
&x_1, x_2 \geq 0
\end{aligned}$$

Introduce appropriate further variables, parameters and constraints as needed and give a modified/extended linear (!) version of that program in which you ensure that

i. if x_1 is strictly positive ($x_1 > 0$), the objective function value decreases by a fixed amount of 3 and if x_2 is strictly positive ($x_2 > 0$), the objective function value decreases by a fixed amount of 4 and that

ii. at most one of the two variables x_1 and x_2 can assume a strictly positive value in any feasible solution, but not both.

Briefly explain the mechanisms of your modeling approach! (6 P.)