Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
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Seat number:

Summer semester 2019
Exam for the lecture
"Operations Research"

Please note:

- The exam consists of **13** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	Sum
Points					

1. **Modeling a mixture problem (20 P.)**

   A company produces different products $p \in \mathcal{P} = \{1, 2, ..., P\}$. The specifications for each product set both a lower and an upper limit on the amount of ingredient i per unit of product p. Denote those lower and upper limits as $a_{p,i}^{low}$ and $a_{p,i}^{up}$, respectively. In order to produce the products, different raw materials $r \in \mathcal{R} = \{1, 2, ..., R\}$ are bought and mixed in the production process. A unit of raw material r contains $b_{r,i}$ units of ingredient i. A unit of raw material r costs c_r. The demand d_p for product p is given and has to be met.

   (a) Formulate a linear program to determine the cost-minimizing quantities $X_{r,p}$ for the production process so that for each product the limits on the ingredients are respected. (8 P.)

   (b) Let a binary variable $Y_r = 1$ if $X_{r,p} > 0$ for any product p, i.e., if any positive quantity of raw material r is bought and used, and $Y_r = 0$ otherwise.

      i. Now assume that if raw material 7 is used, raw materials 8 and 11 must be used. Give the required linear constraint(s) on the variables Y_r! (2 P.)

ii. Now assume that if raw material 15 is not used, both materials 5 and 22 must not be used. Give the required linear constraint(s) on the variables Y_r! (2 P.)

iii. Now assume that if raw material 10 or 13 is used, either material 5 or material 22 must also be used, but not both. Give the required linear constraint(s) on the variables Y_r! (3 P.)

iv. Now assume that if a positive quantity of raw material r is bought, in addition to the unit cost c_r defined above, a fixed order cost f_r is incurred. Give the required additional constraints linking variables Y_r to the variables $X_{r,p}$ and state the new objective function. (2 P.)

v. Give a set of constraints that link binary variables W_r to the variables $X_{r,p}$ such that they force $W_r = 0$ if any $X_{r,p} > 0$, irrespective of the objective function. (3 P.)

2. **Improvement search: (11 P.)**

   (a) The following shows the sequence of directions and step sizes employed by an improving search that began at $\mathbf{y}^{(0)} = (2, 1, 3)$. Compute the sequence of points visited by the search. (2 P.)

$$\Delta\mathbf{y}^{(1)} = (3, 4, 1), \lambda_1 = 3,$$
$$\Delta\mathbf{y}^{(2)} = (5, 2, 1), \lambda_2 = 2$$

(b) Construct an improving direction from the gradient of each of the given objective functions at the point indicated. (6 P.)

i. max $\quad (w_1)^2 + 2w_1w_2^2$ at point $\mathbf{w} = (1, 2)$

ii. min $\quad \ln(w_1) + 3w_2^2$ at point $\mathbf{w} = (1, 200)$

iii. max $\quad 2(w_1)^2 + w_2 w_4 - 3(w_5)^2$ at point $\mathbf{w} = (2, 4, 7, 0, 1)$

(c) The following shows for each of the objective functions from (b) a direction $\Delta\mathbf{w}$. Determine by an appropriate gradient test whether this direction improves on the respective objective function. (3 P.)

i. max $\quad (w_1)^2 + 2w_1 w_2^2$ at point $\mathbf{w} = (1, 2) : \boxed{\Delta\mathbf{w} = (1, -1)}$

ii. min $\ln(w_1) + 3w_2^2$ at point $\mathbf{w} = (1, 200)$: $\boxed{\Delta\mathbf{w} = (-2, 3)}$

iii. max $2(w_1)^2 + w_2 w_4 - 3(w_5)^2$ at point $\mathbf{w} = (2, 4, 7, 0, 1)$: $\boxed{\Delta\mathbf{w} = (0, 2, 3, 12, 2)}$

3. **Branch & Bound (14 P.)**

The following table shows for a binary maximization problem with decision variables $x_1, x_2, x_3 \in \{0, 1\}$ for all combinations of fixed and free, i.e., binary-relaxed, variables the LP relaxation optima $\tilde{x}$ with objective function value $\tilde{v}$.

x_1	x_2	x_3	$\tilde{x}$	$\tilde{v}$
#	#	#	(1.00, 0.38, 1.00)	21.000
#	#	0	(1.00, 0.88, 0.00)	20.000
#	#	1	(0.00, 0.75, 1.00)	21.000
#	0	#	(1.00, 0.00, 1.00)	15.000
#	0	0	(1.00, 0.00, 0.00)	6.000
#	0	1	(1.00, 0.00, 1.00)	15.000
#	1	#	(0.00, 1.00, 0.50)	20.500
#	1	0	(0.67, 1.00, 0.00)	20.000
#	1	1	Infeasible	
0	#	#	(0.00, 0.75, 1.00)	21.000
0	#	0	(0.00, 1.00, 0.00)	16.000
0	#	1	(0.00, 0.75, 1.00)	21.000
0	0	#	(0.00, 0.00, 1.00)	9.000
0	0	0	(0.00, 0.00, 0.00)	0.000
0	0	1	(0.00, 0.00, 1.00)	9.000
0	1	#	(0.00, 1.00, 0.50)	20.500
0	1	0	(0.00, 1.00, 0.00)	16.000
0	1	1	Infeasible	
1	#	#	(1.00, 0.38, 1.00)	21.000
1	#	0	(1.00, 0.88, 0.00)	20.000
1	#	1	(1.00, 0.38, 1.00)	21.000
1	0	#	(1.00, 0.00, 1.00)	15.000
1	0	0	(1.00, 0.00, 0.00)	6.000
1	0	1	(1.00, 0.00, 1.00)	15.000
1	1	#	Infeasible	
1	1	0	Infeasible	
1	1	1	Infeasible	

Determine the optimal solution of the binary maximization problem by applying the Branch&Bound Algorithm 12A with the following specifications:

- Perform a depth-first search!
- When selecting between active candidate problems, break ties in favor of $x_i = 0$, i.e., first create a new candidate by rounding down, and only later by rounding up ($x_i = 1$).
- Number the candidate problems in the sequence you analyze their relaxations.
- Document in your search tree you draw for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.

4. **Optimization Algorithms (15 P.)**

(a) Improving Search Algorithm 3A. (4 P.)

i. Which fundamental property must the feasible set of a problem possess such that the improving search algorithm 3A can be guaranteed to find an optimal solution if one exits? Why?

ii. Which fundamental property must the objective function of a problem possess such that the improving search algorithm 3A can be guaranteed to find an optimal solution if one exists? Why?

(b) Linear Programming. (11 P.)

i. Explain how in Linear Programming the so-called "reduced cost" $\bar{c}_j$ is calculated during any iteration of the simplex algorithm! (2 P.)

ii. What is the relationship between shadow prices and main constraints being active/binding or not? (2 P.)

iii. Explain what a basic solution to a linear program is, how non-basic variables can be distinguished from basic variables in a basic solution, and how you can tell whether a basic solution is a feasible basic solution! (3 P.)

iv. Assume that during a Simplex search for a maximization problem you reached a feasible basic solution. Which condition must hold if this solution is not optimal? Assuming that the current solution is not optimal, explain the steps during the next iteration of the Simplex algorithm! (4 P.)