Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
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Seat number:

Summer semester 2022
Exam for the lecture
“Operations Research”

Please note:

- The exam consists of **13** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice. You may also use an English language dictionary.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	Sum
Points					

1. **Linear Programming** **(12 P.)**

Consider the following incompletely documented Simplex search.

	x_1	x_2	x_3	x_4	x_5	
max **c**	3	9	0	0	0	**b**
	3	3	1	0	0	18
A	3	-6	0	1	0	6
	0	3	0	0	1	12
t=0	N	(1)	B	(2)	B	
$\mathbf{x}^{(0)}$	(3)	0	(4)	6	12	$\mathbf{c} \cdot x^{(0)} = 0$
$\Delta\mathbf{x}$ for x_1	(5)	(6)	-3	(7)	0	$\bar{c_1} = 3$
$\Delta\mathbf{x}$ for x_2	0	(8)	-3	6	-3	$\bar{c_2} = 9$
	–	–	$\frac{18}{-(-3)}$	–	$\frac{12}{-(-3)}$	(9)
t=1	N	B	B	B	N	
$x^{(1)}$	0	4	6	30	0	$\mathbf{c} \cdot x^{(1)} = 36$
$\Delta\mathbf{x}$ for x_1	1	0	-3	-3	0	$\bar{c_1} = 3$
$\Delta\mathbf{x}$ for x_5	0	$-\frac{1}{3}$	1	-2	1	$\bar{c_5} = -3$
	–	–	(10)	$\frac{30}{-(-3)}$	–	(11)
t=2	B	B	N	B	N	
$x^{(2)}$	2	(12)	(13)	24	0	(14)
$\Delta\mathbf{x}$ for x_3	$-\frac{1}{3}$	0	1	1	0	$\bar{c_3} = -1$
$\Delta\mathbf{x}$ for x_5	$\frac{1}{3}$	$-\frac{1}{3}$	0	-3	1	$\bar{c_5} = -2$
					Your result:	

(a) The numbered circles (1) to (14) indicate missing values/entries. Please determine these values/entries and enter them in the table below. (7 P.)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
(8)	**(9)**	**(10)**	**(11)**	**(12)**	**(13)**	**(14)**

(b) How do you interpret the result of the computation? Why? (2 P.)

(c) What can you say about the optimal value of the dual variable that is related to the second constraint? (1 P.)

(d) Assume that a linear program has been solved to optimality. Now consider constraint j of that linear program with slack variable s_j and dual variable v_j. What can you say about the relationship between those two variables? Explain! (2 P.)

2. **Branch & Bound** **(14 P.)**

The following table shows for a binary maximization problem with decision variables $x_1, x_2, x_3 \in \{0, 1\}$ for all combinations of fixed and free, i.e., binary-relaxed, variables the LP relaxation optima $\tilde{x}$ with objective function value $\tilde{v}$.

x_1	x_2	x_3	$\tilde{x}$	$\tilde{v}$
#	#	#	(1.00, 1.00, 0.50)	100.500
#	#	0	(1.00, 1.00, 0.00)	81.000
#	#	1	(0.33, 1.00, 1.00)	100.000
#	0	#	(1.00, 0.00, 1.00)	69.000
#	0	0	(1.00, 0.00, 0.00)	30.000
#	0	1	(1.00, 0.00, 1.00)	69.000
#	1	#	(1.00, 1.00, 0.50)	100.500
#	1	0	(1.00, 1.00, 0.00)	81.000
#	1	1	(0.33, 1.00, 1.00)	100.000
0	#	#	(0.00, 1.00, 1.00)	90.000
0	#	0	(0.00, 1.00, 0.00)	51.000
0	#	1	(0.00, 1.00, 1.00)	90.000
0	0	#	(0.00, 0.00, 1.00)	39.000
0	0	0	(0.00, 0.00, 0.00)	0.000
0	0	1	(0.00, 0.00, 1.00)	39.000
0	1	#	(0.00, 1.00, 1.00)	90.000
0	1	0	(0.00, 1.00, 0.00)	51.000
0	1	1	(0.00, 1.00, 1.00)	90.000
1	#	#	(1.00, 1.00, 0.50)	100.500
1	#	0	(1.00, 1.00, 0.00)	81.000
1	#	1	(1.00, 0.60, 1.00)	99.600
1	0	#	(1.00, 0.00, 1.00)	69.000
1	0	0	(1.00, 0.00, 0.00)	30.000
1	0	1	(1.00, 0.00, 1.00)	69.000
1	1	#	(1.00, 1.00, 0.50)	100.500
1	1	0	(1.00, 1.00, 0.00)	81.000
1	1	1	Infeasible	

Determine the optimal solution of the binary maximization problem by applying the Branch&Bound Algorithm 12A with the following specifications:

- Perform a depth-first search!
- When selecting between active candidate problems, break ties in favor of $x_i = 0$, i.e., first create a new candidate by rounding down, and only later by rounding up ($x_i = 1$).
- Number the candidate problems in the sequence in which you analyze their relaxations.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.

3. **Duality and Benders decomposition** **(9 P.)**

(a) Assume that during a Benders decomposition iteration the dual subproblem BD_2 in iteration $l = 2$ is solved for current value of the single binary variable $y^{(1)} = 3$ from the last solution of the partial master problem BM_1, leading to the following dual subproblem:

$$(BD_2) \quad \begin{array}{lrcl} \max & 6v_1 - 4y^{(1)}v_2 + 3y^{(1)} & = & 6v_1 - 4 \cdot 3 \cdot v_2 + 3 \cdot 3 \\ \text{s.t.} & v_1 - 2v_2 & \leq & 6 \\ & v_1 - 3v_2 & \leq & 5 \\ & v_1, v_2 & \geq & 0 \end{array}$$

Now assume that the optimal solution to that subproblem is $(v_1, v_2) = (8, 1)$.

i. Give the cut that has to be added to the partial master problem. (2 P.)

ii. Explain what kind of cut it is and which effect it has within the Benders decomposition approach! (2 P.)

iii. How is the feasibility of the dual subproblem in a Benders decomposition affected by the outcome of the last solution to the partial master problem? Why? (2 P.)

iv. When does the Benders algorithm applied to a feasible problem terminate and how do we determine the complete solution to the underlying primal problem? (2 P.)

4. **Modeling, cutting stock problem, and delayed column generation** **(25 P.)**

In the so-called cutting stock problem, we have do decide how to cut so-called stock boards of length b into pieces to meet the demand d_i for final board types i. Each final board of board type i has length h_i. In order to minimize the waste, we want to cut the stock boards in such a way that we need to cut as few stock boards as possible.

(a) Give a direct (i.e., compact) model formulation of the cutting stock problem, using the notation presented in the table below. Explain the objective function and each set of constraints! (6 P.)

Sets:	
$n \in \mathcal{N}$	set of (uncut) stock boards, $\mathcal{N} = \{1, ..., N\}$
$i \in \mathcal{I}$	set of required board types, $\mathcal{I} = \{1, ..., I\}$
Parameters:	
d_i	demand for board type i
h_i	length of board type i
b	length of (uncut) stock board
Decision Variables:	
Y_n	binary variable which is equal to 1 if stock board n is used, 0 otherwise
$W_{i,n}$	integer number of boards of type i to be cut from stock board n

(b) What is meant if we say that the model possesses symmetric solutions? Is this good or bad? Why? (3 P.)

(c) Now assume that we want to operate with an indirect modeling approach based on cutting patterns, using the notation presented in the table below. Give the indirect, i.e., pattern-based model formulation of the cutting stock problem and explain the objective function and the constraints. (4 P.)

Sets:	
$i \in \mathcal{I}$	set of final board types, $\mathcal{I} = \{1, \dots, I\}$
$k \in \mathcal{K}$	set of cutting patterns, $\mathcal{K} = \{1, \dots, K\}$
Parameters:	
$a_{i,k}$	number of units of board type i in cutting pattern k
d_i	demand for board type i
Decision Variables:	
X_k	integer number of stock boards to be cut with cutting pattern k

(d) Assume that the set $\mathcal{K}$ of cutting patterns is incomplete in the sense that it does not contain all possible cutting patterns. Is is possible that an optimal solution to the pattern-based model has the **same** objective function value as an optimal solution to the direct (i.e., compact) model formulation? Explain your reasoning! (3 P.)

(e) Assume that we want to generate new cutting patterns via a column generation approach and use the notation presented in the table below.

Sets:	
$i \in \mathcal{I}$	set of final board types, $\mathcal{I} = \{1, \ldots, I\}$
Parameters:	
~~d_i~~	~~demand for board type i~~ b length
h_i	length of board type i
v_i	?????
Decision Variables:	
a_i	integer number of units of board type i in the cutting pattern

Give the subproblem to be solved, explaining the objective function and the constraint! Comment on the origin and meaning of the parameter v_i! How do we call problems with this structure? (7 P.)

(f) Assume that we want to solve the linear programming relaxation of the pattern-based model formulation of the cutting stock problem via column generation. When does the column generation process stop? Why? (2 P.)