Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
Prof. Dr. Stefan Helber

Seat number:

Winter semester 2019-2020
Exam for the lecture
"Operations Research"

Please note:

- The exam consists of **14** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice. You may also use an English language dictionary.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	5	6	Sum
Points							

1. **Elements of modeling (10 P.)**

   Assume that in a cost-minimizing production planning model, the real-valued non-negative decision variable X_p denotes the production quantity of product p and that c_p denotes the respective unit production cost. Let u_p denote an upper bound on X_p. Let a binary variable Y_p equal 1 if $X_p > 0$ for any product p, and 0 otherwise. Denote with f_p the fixed cost which is incurred for any positive production quantity X_p.

   (a) Using this notation, give and explain a constraint that forces Y_p to equal 1 for any positive value of X_p for each $p \in \mathcal{P}$. Why is it important that we know to be minimizing cost in our objective function? (3 P.)

   (b) Now assume that if product 8 is produced, products 7 and 11 cannot be produced. Give the required linear constraint(s) on the variables Y_p! (2 P.)

(c) Now assume that if product 8 is not produced, products 7 and 11 must be produced. Give the required linear constraint(s) on the variables Y_p! (2 P.)

(d) Now assume that if product 8 is produced, either products 3 and 10 must be produced, but not both. Give the required linear constraint(s) on the variables Y_p! (3 P.)

2. **Improvement search: (10 P.)**

   (a) The following shows the sequence of directions and step sizes employed by an improving search that began at $\mathbf{y}^{(0)} = (2, 3, 1)$. Compute the sequence of points visited by the search. (2 P.)

$$\Delta\mathbf{y}^{(1)} = (3, 5, 2), \lambda_1 = 3,$$
$$\Delta\mathbf{y}^{(2)} = (2, 1, 3), \lambda_2 = 2$$

(b) Assume we are maximizing the function $f(w_1, w_2) = (w_1)^2 + 2w_1w_2^2$.

i. Construct at point $\mathbf{w} = (w_1, w_2) = (2, 2)$ an improving direction from the gradient of this function. (2 P.)

ii. Determine by an appropriate gradient test whether at point $\mathbf{w} = (w_1, w_2) = (2, 2)$ the direction $\Delta\mathbf{w} = (-3, 2)$ improves on the objective function. (2 P.)

(c) Which fundamental property must the feasible set of a problem possess such that the improving search algorithm 3A can be guaranteed to find an optimal solution if one exits? Why? (2 P.)

(d) Which fundamental property must the objective function of a problem possess such that the improving search algorithm 3A can be guaranteed to find an optimal solution if one exists? Why? (2 P.)

3. **Branch & Bound** **(14 P.)**

The following table shows for a binary maximization problem with decision variables $x_1, x_2, x_3 \in \{0, 1\}$ for all combinations of fixed and free, i.e., binary-relaxed, variables the LP relaxation optima $\tilde{x}$ with objective function value $\tilde{v}$.

x_1	x_2	x_3	$\tilde{x}$	$\tilde{v}$
#	#	#	(1.00, 0.50, 1.00)	100.500
#	#	0	(1.00, 1.00, 0.00)	69.000
#	#	1	(1.00, 0.50, 1.00)	100.500
#	0	#	(1.00, 0.00, 1.00)	81.000
#	0	0	(1.00, 0.00, 0.00)	30.000
#	0	1	(1.00, 0.00, 1.00)	81.000
#	1	#	(0.33, 1.00, 1.00)	100.000
#	1	0	(1.00, 1.00, 0.00)	69.000
#	1	1	(0.33, 1.00, 1.00)	100.000
0	#	#	(0.00, 1.00, 1.00)	90.000
0	#	0	(0.00, 1.00, 0.00)	39.000
0	#	1	(0.00, 1.00, 1.00)	90.000
0	0	#	(0.00, 0.00, 1.00)	51.000
0	0	0	(0.00, 0.00, 0.00)	0.000
0	0	1	(0.00, 0.00, 1.00)	51.000
0	1	#	(0.00, 1.00, 1.00)	90.000
0	1	0	(0.00, 1.00, 0.00)	39.000
0	1	1	(0.00, 1.00, 1.00)	90.000
1	#	#	(1.00, 0.50, 1.00)	100.500
1	#	0	(1.00, 1.00, 0.00)	69.000
1	#	1	(1.00, 0.50, 1.00)	100.500
1	0	#	(1.00, 0.00, 1.00)	81.000
1	0	0	(1.00, 0.00, 0.00)	30.000
1	0	1	(1.00, 0.00, 1.00)	81.000
1	1	#	(1.00, 1.00, 0.60)	99.600
1	1	0	(1.00, 1.00, 0.00)	69.000
1	1	1	Infeasible	

Determine the optimal solution of the binary maximization problem by applying the Branch&Bound Algorithm 12A with the following specifications:

- Perform a depth-first search!
- When selecting between active candidate problems, break ties in favor of $x_i = 0$, i.e., first create a new candidate by rounding down, and only later by rounding up ($x_i = 1$).
- Number the candidate problems in the sequence in which you analyze their relaxations.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.

4. **Linear Programming** **(12 P.)**

Consider the following incompletely documented Simplex search.

	x_1	x_2	x_3	x_4	x_5	
max **c**	3	9	0	0	0	**b**
	3	3	1	0	0	18
A	3	-6	0	1	0	6
	0	3	0	0	1	12
t=0	N	(1)	B	(2)	B	
$\mathbf{x}^{(0)}$	(3)	0	(4)	6	12	$\mathbf{c} \cdot x^{(0)} = 0$
$\Delta\mathbf{x}$ for x_1	(5)	(6)	-3	(7)	0	$\bar{c_1} = 3$
$\Delta\mathbf{x}$ for x_2	0	(8)	-3	6	-3	$\bar{c_2} = 9$
	–	–	$\frac{18}{-(-3)}$	–	$\frac{12}{-(-3)}$	(9)
t=1	N	B	B	B	N	
$x^{(1)}$	0	4	6	30	0	$\mathbf{c} \cdot x^{(1)} = 36$
$\Delta\mathbf{x}$ for x_1	1	0	-3	-3	0	$\bar{c_1} = 3$
$\Delta\mathbf{x}$ for x_5	0	$-\frac{1}{3}$	1	-2	1	$\bar{c_5} = -3$
	–	–	(10)	$\frac{30}{-(-3)}$	–	(11)
t=2	B	B	N	B	N	
$x^{(2)}$	2	(12)	(13)	24	0	(14)
$\Delta\mathbf{x}$ for x_3	$-\frac{1}{3}$	0	1	1	0	$\bar{c_3} = -1$
$\Delta\mathbf{x}$ for x_5	$\frac{1}{3}$	$-\frac{1}{3}$	0	-3	1	$\bar{c_5} = -2$
					Your result:	

(a) The numbered circles (1) to (14) indicate missing values/entries. Please determine these values/entries and enter them in the table below. (7 P.)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
(8)	(9)	(10)	(11)	(12)	(13)	(14)

(b) How do you interpret the result of the computation? Why? (2 P.)

(c) What can you say about the optimal value of the dual variable that is related to the third constraint? (1 P.)

(d) Assume that a linear program has been solved to optimality. Now consider constraint j of that linear program with slack variable s_j and dual variable v_j. What can you say about the relationship between those two variables? Explain! (2 P.)

5. **Benders decomposition** **(8 P.)**

(a) Assume that during a Benders decomposition iteration the dual subproblem BD_2 in iteration $l = 2$ is solved for current value of the single binary variable $y^{(1)} = 3$ from the last solution of the partial master problem BM_1, leading to the following dual subproblem:

$$
(BD_2) \quad
\begin{array}{lrcl}
\max & 6v_1 - 4y^{(1)}v_2 + 3y^{(1)} & = & 6v_1 - 4 \cdot 3 \cdot v_2 + 3 \cdot 3 \\
\text{s.t.} & v_1 - 2v_2 & \leq & 6 \\
 & v_1 - 3v_2 & \leq & 5 \\
 & v_1, v_2 & \geq & 0
\end{array}
$$

Now assume that the optimal solution to that subproblem is $(v_1, v_2) = (8, 1)$.

i. Give the cut that has to be added to the partial master problem. (2 P.)

ii. Explain what kind of cut it is and which effect it has within the Benders decomposition approach! (2 P.)

iii. How is the feasibility of the dual subproblem in a Benders decomposition affected by the outcome of the last solution to the partial master problem? Why? (2 P.)

iv. When does the Benders algorithm applied to a feasible problem terminate and how do we determine the complete solution to the underlying primal problem? (2 P.)

6. **Delayed column generation** **(6 P.)**

(a) What are the potential disadvantages of a direct modeling approach for a mixed-integer or combinatorial optimization problem (as opposed to a delayed column generation approach)? (2 P.)

(b) How is the subproblem in a delayed column generation approach affected by the last solution of the master problem? (1 P.)

(c) How is the master problem in a delayed column generation approach affected by the last solution of the subproblem? (1 P.)

(d) Assume a column generation approach is applied to a minimization problem. When does the algorithm stop? Which important property does the objective function value of the final master problem possess? (2 P.)