Leibniz Universität Hannover
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Seat number:

Winter semester 2021-2022
Exam for the lecture
"Operations Research"

Please note:

- The exam consists of **14** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice. You may also use an English language dictionary.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	5	Sum
Points						

1. **Improvement search: (14 P.)**

   (a) The following shows the sequence of directions and step sizes employed by an improving search that began at $\mathbf{y}^{(0)} = (2, 4, 2)$. Compute the sequence of points visited by the search. (2 P.)

$$\Delta\mathbf{y}^{(1)} = (3, 2, 2), \lambda_1 = 4,$$
$$\Delta\mathbf{y}^{(2)} = (1, 3, 2), \lambda_2 = 3$$

(b) Assume we are maximizing the function $f(w_1, w_2) = 2(w_1)^2 + 3w_1w_2^2$.

i. Construct at point $\mathbf{w} = (w_1, w_2) = (2, 2)$ an improving direction from the gradient of this function. (2 P.)

ii. Determine by an appropriate gradient test whether at point $\mathbf{w} = (w_1, w_2) = (2, 3)$ the direction $\Delta\mathbf{w} = (-2, 1)$ improves on the objective function. (2 P.)

(c) Draw an example of a convex two-dimensional feasible set and explain why the feasible set is convex! (2 P.)

(d) Draw an example of a non-convex two-dimensional feasible set and explain why the feasible set is not convex! (2 P.)

(e) How does the (non-)convexity of the feasible set affect the applicability of the Improving Search Algorithm if we want to solve a problem in continuous variables to optimality? Explain! (2 P.)

(f) Which fundamental property must the objective function of a problem possess such that the improving search algorithm 3A can be guaranteed to find an optimal solution if one exists? Why? (2 P.)

2. **Branch & Bound** **(14 P.)**

The following table shows for a binary maximization problem with decision variables $x_1, x_2, x_3 \in \{0, 1\}$ for all combinations of fixed and free, i.e., binary-relaxed, variables the LP relaxation optima $\tilde{x}$ with objective function value $\tilde{v}$.

x_1	x_2	x_3	$\tilde{x}$	$\tilde{v}$
#	#	#	(1.00, 1.00, 0.50)	100.500
#	#	0	(1.00, 1.00, 0.00)	81.000
#	#	1	(0.33, 1.00, 1.00)	100.000
#	0	#	(1.00, 0.00, 1.00)	69.000
#	0	0	(1.00, 0.00, 0.00)	30.000
#	0	1	(1.00, 0.00, 1.00)	69.000
#	1	#	(1.00, 1.00, 0.50)	100.500
#	1	0	(1.00, 1.00, 0.00)	81.000
#	1	1	(0.33, 1.00, 1.00)	100.000
0	#	#	(0.00, 1.00, 1.00)	90.000
0	#	0	(0.00, 1.00, 0.00)	51.000
0	#	1	(0.00, 1.00, 1.00)	90.000
0	0	#	(0.00, 0.00, 1.00)	39.000
0	0	0	(0.00, 0.00, 0.00)	0.000
0	0	1	(0.00, 0.00, 1.00)	39.000
0	1	#	(0.00, 1.00, 1.00)	90.000
0	1	0	(0.00, 1.00, 0.00)	51.000
0	1	1	(0.00, 1.00, 1.00)	90.000
1	#	#	(1.00, 1.00, 0.50)	100.500
1	#	0	(1.00, 1.00, 0.00)	81.000
1	#	1	(1.00, 0.60, 1.00)	99.600
1	0	#	(1.00, 0.00, 1.00)	69.000
1	0	0	(1.00, 0.00, 0.00)	30.000
1	0	1	(1.00, 0.00, 1.00)	69.000
1	1	#	(1.00, 1.00, 0.50)	100.500
1	1	0	(1.00, 1.00, 0.00)	81.000
1	1	1	Infeasible	

Determine the optimal solution of the binary maximization problem by applying the Branch&Bound Algorithm 12A with the following specifications:

- Perform a depth-first search!
- When selecting between active candidate problems, break ties in favor of $x_i = 0$, i.e., first create a new candidate by rounding down, and only later by rounding up ($x_i = 1$).
- Number the candidate problems in the sequence in which you analyze their relaxations.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.

3. **Linear Programming** **(12 P.)**

Consider the following incompletely documented Simplex search.

	x_1	x_2	x_3	x_4	x_5	
max **c**	3	9	0	0	0	**b**
	3	3	1	0	0	18
A	3	-6	0	1	0	6
	0	3	0	0	1	12
t=0	N	(1)	B	(2)	B	
$\mathbf{x}^{(0)}$	(3)	0	(4)	6	12	$\mathbf{c} \cdot x^{(0)} = 0$
$\Delta\mathbf{x}$ for x_1	(5)	(6)	-3	(7)	0	$\bar{c_1} = 3$
$\Delta\mathbf{x}$ for x_2	0	(8)	-3	6	-3	$\bar{c_2} = 9$
	–	–	$\frac{18}{-(-3)}$	–	$\frac{12}{-(-3)}$	(9)
t=1	N	B	B	B	N	
$x^{(1)}$	0	4	6	30	0	$\mathbf{c} \cdot x^{(1)} = 36$
$\Delta\mathbf{x}$ for x_1	1	0	-3	-3	0	$\bar{c_1} = 3$
$\Delta\mathbf{x}$ for x_5	0	$-\frac{1}{3}$	1	-2	1	$\bar{c_5} = -3$
	–	–	(10)	$\frac{30}{-(-3)}$	–	(11)
t=2	B	B	N	B	N	
$x^{(2)}$	2	(12)	(13)	24	0	(14)
$\Delta\mathbf{x}$ for x_3	$-\frac{1}{3}$	0	1	1	0	$\bar{c_3} = -1$
$\Delta\mathbf{x}$ for x_5	$\frac{1}{3}$	$-\frac{1}{3}$	0	-3	1	$\bar{c_5} = -2$
					Your result:	

(a) The numbered circles (1) to (14) indicate missing values/entries. Please determine these values/entries and enter them in the table below. (7 P.)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
(8)	(9)	(10)	(11)	(12)	(13)	(14)

(b) How do you interpret the result of the computation? Why? (2 P.)

(c) What can you say about the optimal value of the dual variable that is related to the third constraint? (1 P.)

(d) Assume that a linear program has been solved to optimality. Now consider constraint j of that linear program with slack variable s_j and dual variable v_j. What can you say about the relationship between those two variables? Explain! (2 P.)

4. **Benders decomposition** **(8 P.)**

(a) Assume that during a Benders decomposition iteration the dual subproblem BD_2 in iteration $l = 2$ is solved for current value of the single binary variable $y^{(1)} = 3$ from the last solution of the partial master problem BM_1, leading to the following dual subproblem:

$$
\begin{array}{llrcl}
 & \max & 6v_1 - 4y^{(1)}v_2 + 3y^{(1)} & = & 6v_1 - 4 \cdot 3 \cdot v_2 + 3 \cdot 3 \\
(BD_2) & \text{s.t.} & v_1 - 2v_2 & \leq & 6 \\
 & & v_1 - 3v_2 & \leq & 5 \\
 & & v_1, v_2 & \geq & 0
\end{array}
$$

Now assume that the optimal solution to that subproblem is $(v_1, v_2) = (8, 1)$.

i. Give the cut that has to be added to the partial master problem. (2 P.)

ii. Explain what kind of cut it is and which effect it has within the Benders decomposition approach! (2 P.)

iii. How is the feasibility of the dual subproblem in a Benders decomposition affected by the outcome of the last solution to the partial master problem? Why? (2 P.)

iv. When does the Benders algorithm applied to a feasible problem terminate and how do we determine the complete solution to the underlying primal problem? (2 P.)

5. **Delayed column generation and the cutting stock problem** **(12 P.)**

(a) What are the potential disadvantages of a direct modeling approach for a mixed-integer or combinatorial optimization problem (as opposed to a delayed column generation approach)? (2 P.)

(b) Assume a column generation approach is applied to a minimization problem. When does the algorithm stop? (1 P.)

(c) Assume that we want to use a column generation approach to solve a cutting stock problem. The raw (uncut) boards have length $b = 150$. The length and demand for the three final board types are as follows:

i	length h_i	demand d_i
1	55	8
2	45	6
3	35	7

Let $a_{i,k}$ denote the number of final boards of type i to be cut out a board in cutting pattern k of the master problem. Assume that in Step 0 of the column generation algorithm we have an initial collection of columns, i.e., cutting patterns, $\mathcal{K}_0 = \{1, 2, 3\}$ with parameters $a_{i,k}$ according to the following matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

i. Write down the (relaxed) master problem of the cutting stock problem for the currently given set of cutting patterns $\mathcal{K}_0$. (2 P.)

ii. Determine (by inspection) the optimal solution of the relaxed master problem and its objective function value. (2 P.)

iii. Determine (again by inspection) the values of the dual variables for the optimal solution of the relaxed master problem. (1 P.)

iv. Given the values of the dual variables stemming from the optimal solution of the relaxed master problem, now write down the subproblem! Explain how the solution of that subproblem is being used in the column generation algorithm! (4 P.)