Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
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Seat number:

Winter semester 2022-2023
Exam for the lecture
"Operations Research"

Please note:

- The exam consists of **11** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice. You may also use an English language dictionary.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	5	Sum
Points						

1. **Improvement search: (11 P.)**

   (a) Assume we are minimizing the function $f(w_1, w_2) = 2w_1^2 w_2^3$.

      i. Construct at point $\mathbf{w} = (w_1, w_2) = (2, 2)$ an improving direction from the gradient of this function. (2 P.)

      ii. Determine by an appropriate gradient test whether at point $\mathbf{w} = (w_1, w_2) = (1, 2)$ the direction $\Delta\mathbf{w} = (-2, 1)$ improves on the objective function. (2 P.)

(b) Explain briefly what characterizes in an improving search an "active constraint" and what a "feasible direction". (3 P.)

(c) Which fundamental properties must the objective function and the feasible set of a problem possess such that the improving search algorithm 3A can be guaranteed to find an optimal solution if one exists? Why? (4 P.)

2. **Branch & Bound** **(15 P.)**

Consider the following integer linear program with a **minimization** objective function:

$$
\begin{aligned}
&\text{Min } 2x_1 + 3x_2 \\
&\text{s.t.} \\
&7x_1 + 3x_2 \geq 22 \\
&2x_1 + 8x_2 \geq 16 \\
&x_i \in \{0, 1, 2, 3, 4, ...\}, \qquad\qquad i \in \{1, 2\}
\end{aligned}
$$

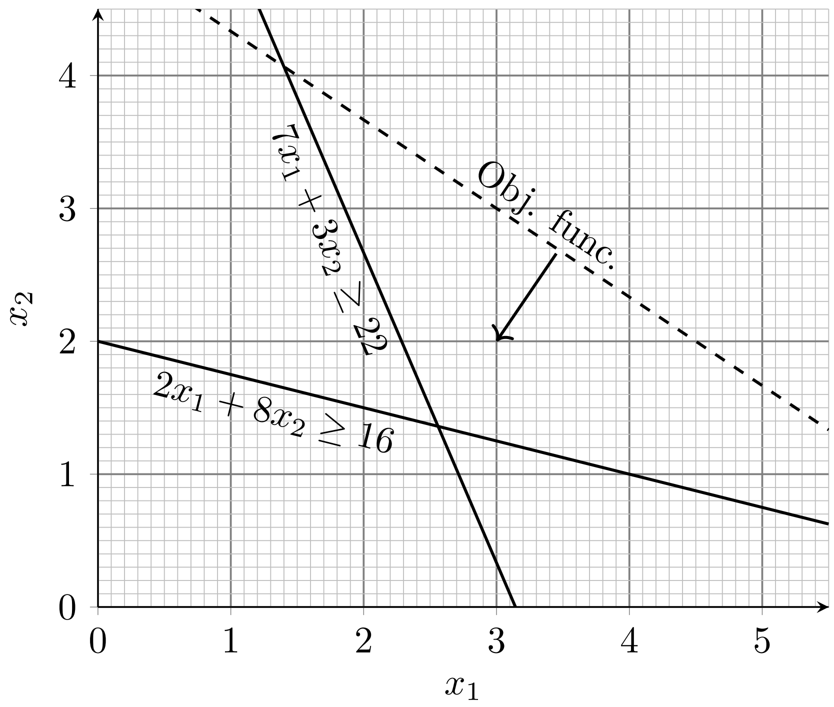
Determine the optimal solution of this integer **minimization** problem by applying a branch&bound algorithm according to the following specifications:

- Perform a depth-first search!
- Start with incumbent solution ($x_1 = 5,\ x_2 = 5$).
- If in the linear programming relaxation of a candidate problem both variables x_1 and x_2 should be fractional, break ties in favor of the first variable, i.e., branch on x_1.
- When branching on a fractional variable x_i by creating two new problems, create the new problem to be examined next in the depth-first search by rounding up and the other new problem (to be examined later in the depth-first search) by rounding down.
- Number the problems according to the sequence in which you determine their relaxations and make branching or bounding decisions in the decision tree based on the solution of the candidate problem.

Hints and tasks:

- You can determine the values of relaxed variables x_1 and x_2 in a relaxation of a candidate problem to a sufficient degree of accuracy by modifying and reading from the figure on the following page. On this basis you can compute to a sufficient degree of accuracy the objective function values as well.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.

The following figure (see next page) shows the visualization of the objective function and the constraints:



$7x_1 + 3x_2 \geq 22$
Obj. func.
$2x_1 + 8x_2 \geq 16$
x_1
x_2
0
1
2
3
4
5

3. **Linear Programming** **(20 P.)**

Consider the following linear program (LP), denoted as the "primal" problem:

$$
\begin{aligned}
\text{Max } & 11x_1 + 10x_2 \\
\text{subject to} & \\
x_1 \quad\quad & \leq 3 \\
x_2 & \leq 4 \\
x_1 + x_2 & \leq 8 \\
x_1, x_2 & \geq 0
\end{aligned}
$$

(a) Determine by inspection
   i. the optimal solution to the problem, (2 P.)
   ii. the objective function value for the optimal solution, (1 P.)
   iii. the values of the dual variables in the optimal solution. (2 P.)

(b) Using your results, show that for all three constraints the primal complementary slackness condition holds! (3. P)

(c) Determine the dual program corresponding to this primal LP! (2 P.)

(d) Give the standard form of the primal LP! (2 P.)

(e) Does the standard form version of this primal LP possess a variable which can never be non-basic? Explain! (1 P.)

(f) Assume that in a solution to the standard form of this primal LP, variables x_1 and x_2 are non-basic.

i. Give the complete solution for this case! (2 P.)

ii. Set up and solve the system of equations to determine the direction vector of rudimentary Simplex algorithm 3A to **make non-basic variable x_1 basic** and determine the associated unit change $\bar{c}_1$ (also known as “reduced cost”)! (3 P.)

iii. Assume that we decided to actually make x_1 basic, using the just determined direction vector. What would be the step size and what the next solution vector? (2 P.)

4. **Delayed column generation** **(6 P.)**

(a) What is the respective role of the master problem and the subproblem in a column generation approach and how do these two problems interact? (4 P.)

(b) What is the potential role of a column generation approach embedded in a branch-and-bound algorithm to solve a mixed-integer linear program? What is the name of the resulting type of algorithm? (2 P.)

5. **Modeling:** **(8 P.)**

Consider the following linear program:

$$\begin{aligned}
&\text{Maximize } 3x_1 + x_2 \\
&\text{subject to} \\
&6x_1 + 4x_2 \leq 24 \\
&x_1, x_2 \geq 0
\end{aligned}$$

Introduce appropriate further variables, parameters and constraints as needed and give a modified/extended linear (!) version of that program in which you ensure that

(a) if x_1 is strictly positive ($x_1 > 0$), the objective function value decreases by a fixed amount of 1 and if x_2 is strictly positive ($x_2 > 0$), the objective function value decreases by a fixed amount of 2 and that

(b) at most one of the two variables x_1 and x_2 can assume a strictly positive value in any feasible solution, but not both.

Briefly explain the mechanisms of your modeling approach!