Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
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Seat number:

Winter semester 2023-2024
Exam for the lecture
"Operations Research"

Please note:

- The exam consists of **12** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice. You may also use an English language dictionary.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	5	Sum
Points						

1. **Improvement search: (10 P.)**

   (a) Assume we are minimizing the function $f(w_1, w_2) = 2w_1w_2^2$.

      i. Construct at point $\mathbf{w} = (w_1, w_2) = (2, 3)$ an improving direction from the gradient of this function. (2 P.)

      ii. Determine by an appropriate gradient test whether at point $\mathbf{w} = (w_1, w_2) = (1, 2)$ the direction $\Delta\mathbf{w} = (-2, 1)$ improves on the objective function. (2 P.)

(b) Explain briefly what characterizes in an improving search an "active constraint" and what a "feasible direction". (2 P.)

(c) Which fundamental properties must the objective function and the feasible set of a problem possess such that the improving search algorithm 3A can be guaranteed to find an optimal solution if one exists? Why? (4 P.)

2. **Branch & Bound** **(15 P.)**

The following table shows for a binary maximization problem with decision variables $x_1, x_2, x_3 \in \{0, 1\}$ for all combinations of fixed and free, i.e., binary-relaxed, variables the LP relaxation optima $\tilde{x}$ with objective function value $\tilde{v}$.

x_1	x_2	x_3	$\tilde{x}$	$\tilde{v}$
#	#	#	(1.00, 0.50, 1.00)	100.500
#	#	0	(1.00, 1.00, 0.00)	69.000
#	#	1	(1.00, 0.50, 1.00)	100.500
#	0	#	(1.00, 0.00, 1.00)	81.000
#	0	0	(1.00, 0.00, 0.00)	30.000
#	0	1	(1.00, 0.00, 1.00)	81.000
#	1	#	(0.33, 1.00, 1.00)	100.000
#	1	0	(1.00, 1.00, 0.00)	69.000
#	1	1	(0.33, 1.00, 1.00)	100.000
0	#	#	(0.00, 1.00, 1.00)	90.000
0	#	0	(0.00, 1.00, 0.00)	39.000
0	#	1	(0.00, 1.00, 1.00)	90.000
0	0	#	(0.00, 0.00, 1.00)	51.000
0	0	0	(0.00, 0.00, 0.00)	0.000
0	0	1	(0.00, 0.00, 1.00)	51.000
0	1	#	(0.00, 1.00, 1.00)	90.000
0	1	0	(0.00, 1.00, 0.00)	39.000
0	1	1	(0.00, 1.00, 1.00)	90.000
1	#	#	(1.00, 0.50, 1.00)	100.500
1	#	0	(1.00, 1.00, 0.00)	69.000
1	#	1	(1.00, 0.50, 1.00)	100.500
1	0	#	(1.00, 0.00, 1.00)	81.000
1	0	0	(1.00, 0.00, 0.00)	30.000
1	0	1	(1.00, 0.00, 1.00)	81.000
1	1	#	(1.00, 1.00, 0.60)	99.600
1	1	0	(1.00, 1.00, 0.00)	69.000
1	1	1	Infeasible	

Determine the optimal solution of the binary maximization problem by applying the Branch&Bound Algorithm 12A with the following specifications:

- Perform a depth-first search!
- When selecting between active candidate problems, break ties in favor of $x_i = 1$, i.e., first create a new candidate by rounding up, and only later by rounding down ($x_i = 0$).
- Number the candidate problems in the sequence in which you analyze their relaxations.
- Document in your search tree for each candidate
  - the vector of values of decision variables in that relaxation,
  - the objective function value for that relaxation,
  - if you should have found a new incumbent solution, its objective function value, and
  - if the node of the branch&bound search tree can be terminated, why it can be terminated.
- Give the overall optimal solution and the optimal objective function value when the algorithm has terminated.

3. **Linear Programming** **(14 P.)**

Consider the following linear program (LP), denoted as the "primal" problem:

$$
\begin{aligned}
&\text{Max } 40x_1 + 50x_2 \\
&\text{subject to} \\
&x_1 \qquad\quad \leq 30 \\
&\qquad\quad x_2 \leq 20 \\
&x_1 + x_2 \leq 10 \\
&x_1, x_2 \geq 0
\end{aligned}
$$

(a) Determine by inspection

  i. the optimal solution to the problem, (2 P.)

  ii. the objective function value for the optimal solution, (1 P.)

  iii. the values of the dual variables in the optimal solution. (2 P.)

(b) Using your results, show that for all three constraints the primal complementary slackness condition holds! (3. P)

(c) Determine the dual program corresponding to the primal LP! (2 P.)

(d) Give the standard form of the primal LP! (2 P.)

(e) Does the standard form version of the primal LP possess variables which can never be non-basic? If so, give those variable(s). In either case, justify your answer! (2 P.)

4. **Column generation** **(3 P.)**

Assume you are using a column generation algorithm to solve a linear program with a **minimization** (!!) objective function. When does the process or generating new columns or solution patterns stop? Explain!

5. **Models and Modeling:** **(18 P.)**

(a) Write down algebraically an example of a linear programming model with two decision variables x_1 and x_2 which is feasible and possesses infinitely many optimal solutions with respect to those variables x_1 and x_2. Briefly explain which of the solutions can be found by a simplex algorithm! (4 P.)

(b) Draw graphically an example of a linear program with two decision variables x_1 and x_2 in which multiple different (!) basic solutions of the model in standard form correspond to the same optimal solution vector $x = (x_1^*, x_2^*)$. (4 P.)

(c) Write down an example of a small or even tiny feasible non-linear program and give a hint as to why it is non-linear! (4 P.)

(d) Consider the following binary program:

$$\text{Minimize } 10x_1 + 10x_2 + 10x_3 + 10x_4$$
$$\text{subject to}$$
$$40x_1 + 50x_2 + 60x_3 + 70x_4 \geq 130$$
$$x_1, x_2, x_3, x_4 \in \{0, 1\}$$

For this program

i. determine by inspection the optimal solution and give the objective function value for this solution, (2 P.)

ii. determine by inspection the optimal solution of the linear programming relaxation and give the objective function value for this linear programming relaxation, (2 P.)

iii. and explain how and why a Branch&Bound algorithm to solve such a binary program would proceed in the root node of the search tree, given this optimal solution of the linear programming relaxation of the original problem. (2 P.)