Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
Prof. Dr. Stefan Helber

Seat number:

Weinter semester 2024-2025
Exam for the lecture
"Operations Research"

Please note:

- The exam consists of **13** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice. You may also use an English language dictionary.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	5	6	Sum
Points							

1. **Improvement search: (10 P.)**

   We consider real-valued variables w_1 and w_2.

   (a) Assume we are minimizing the function $f(w_1, w_2) = w_1^2 w_2^3$.

      i. Construct at point $\mathbf{w} = (w_1, w_2) = (2, 1)$ an improving direction from the gradient of this function. (2 P.)

      ii. Determine by an appropriate gradient test whether at point $\mathbf{w} = (w_1, w_2) = (2, 2)$ the direction $\Delta\mathbf{w} = (-2, 1)$ improves on the objective function. (2 P.)

(b) Explain briefly what characterizes in an improving search an "active constraint" and what a "feasible direction". (2 P.)

(c) Draw an example of a feasible set for two continuous variables w_1 and w_2 where an improving search algorithm cannot be guaranteed to find an optimal solution due to this feasible set and explain your reasoning! (2 P.)

(d) Consider the case of an *unlimited* feasible set over two continuous variables $-\infty \leq w_1 \leq \infty$ and $-\infty \leq w_2 \leq \infty$. Now draw an example of the contours of an objective function over this feasible set for which an improving search algorithm cannot be guaranteed to find an optimal solution due the specific shape of the objective function and explain your reasoning! (2 P.)

2. **Branch & Bound** **(15 P.)**

Consider the following integer linear program with a **maximization** objective function:

$$\begin{aligned}
&\text{Max } x_1 + x_2 \\
&\text{s.t.} \\
&2x_1 + 12x_2 \leq 40 \\
&10x_1 + 2x_2 \leq 48 \\
&x_i \in \{0, 1, 2, 3, 4, ...\}, \qquad\qquad i \in \{1, 2\}
\end{aligned}$$

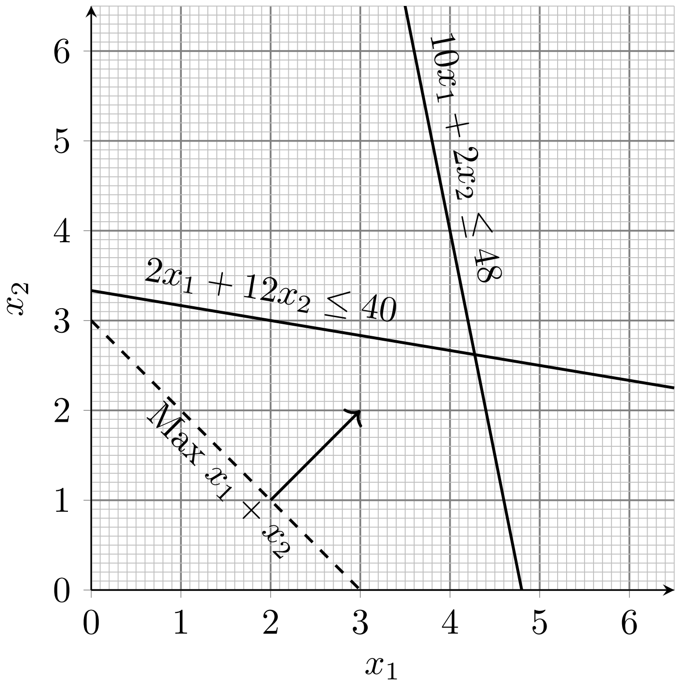
Determine the optimal solution of this integer **maximization** problem by applying a branch&bound algorithm according to the the following specifications:

- Perform a depth-first search!
- Start with incumbent solution ($x_1 = 0,\ x_2 = 0$).
- If in the linear programming relaxation of a candidate problem both variables x_1 and x_2 should be fractional, break ties in favor of the first variable, i.e., branch on x_1.
- When branching on a fractional variable x_i by creating two new problems, create the new problem to be examined next in the depth-first search by rounding up and the other new problem (to be examined later in the depth-first search) by rounding down.
- Number the problems according to the sequence in which you determine their relaxations and make branching or bounding decisions in the decision tree based on the solution of the candidate problem.

Hints and tasks:

- You can determine the values of relaxed variables x_1 and x_2 in a relaxation of a candidate problem to a sufficient degree of accuracy by modifying and reading from the figure on the following page. On this basis you can compute to a sufficient degree of accuracy the objective function values as well.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.

The following figure (see next page) shows the visualization of the objective function and the constraints:



$10x_1 + 2x_2 \le 48$
$2x_1 + 12x_2 \le 40$
Max $x_1 \times x_2$
x_1
x_2
0
1
2
3
4
5
6

3. **Linear Programming** **(15 P.)**

Consider the following linear program (LP), denoted as the "primal" problem:

$$
\begin{aligned}
\text{Min } & 3x_1 + 5x_2 \\
\text{subject to} & \\
x_1 \quad\;\; & \geq 6 \\
x_2 & \geq 4 \\
x_1 + x_2 & \geq 2 \\
x_1, x_2 & \geq 0
\end{aligned}
$$

(a) Determine by inspection for this primal problem

   i. the optimal solution to the problem, (2 P.)

   ii. the objective function value for the optimal solution, (1 P.)

   iii. the values of its dual variables in the optimal solution. (2 P.)

(b) Using your results, show that for all three constraints the primal complementary slackness condition holds! (3. P)

(c) Determine the dual program corresponding to the primal LP! (2 P.)

(d) Determine by inspection for this dual problem

i. the optimal solution to the problem, (2 P.)

ii. the objective function value for the optimal solution, (1 P.)

iii. the values of its dual variables in the optimal solution. (2 P.)

4. **Integer programming** **(7 P.)**

(a) Presolving: Assume Y has to be **integer** and a constraint of a MIP model is

$$Y \geq \frac{11}{10}$$

Can this constraint be tightened during the presolve process? If so, give the tightened constraint, if not, explain why the constraint cannot be tightened! (2 P.)

(b) Adding cuts: Consider the following binary knapsack problem:

$$\begin{aligned} \text{Max } & 10X_1 + 20X_2 + 30X_3 + 40X_4 \\ & \text{subject to} \\ & 4X_1 + 5X_2 + 4X_3 + 10000X_4 \leq 9 \\ & X_1, X_2, X_3, X_4 \in \{0, 1\} \end{aligned}$$

Does the problem possess **minimal** knapsack cover cuts? If so, give one such minimal knapsack cover cut, if not, explain why such a minimal knapsack cover cut does not exist here! (3 P.)

(c) Why is it that mixed-integer programming solvers try to tighten constraints during presolve and to add cuts to the model? (2 P.)

5. **Branch-and-Price** **(7 P.)**

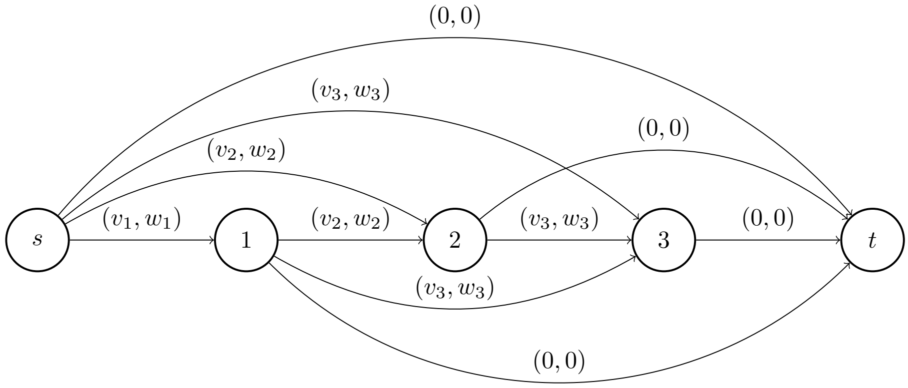
(a) Why can it be attractive to solve a mixed-integer program via a branch-and-price approach, i.e., pattern-based, as opposed to solving the original compact model directly via a branch-and-bound approach in which linear programming relaxations are determined via some simplex algorithm? Hint: Think of dual bounds in either case! (3 P.)

(b) Assume that a column generation approach for a cutting stock problem leads to a fractional solution in terms of the master problem variables representing the extent to which the generated cutting patterns ("columns") are being used. Which problem will occur if you now naively branch on one of the fractional master variables? Explain briefly one possible way to overcome this problem! (4 P.)

6. **Longest-path problem and the label setting algorithm** **(6 P.)**

Consider the following knapsack problem with knapsack capacity $b = 13$:

Item i	1	2	3
Value ("Distance") v_i	10	**-5**	6
Weight w_i	11	1	8



We can interpret the problem as a longest-path problem with a resource constraint. It has been solved via the label setting algorithm treated in class! Below you find the entire solution protocol:

Node	Label l	Pred. label p_l	Cum. value cv_l	Rem. cap. rc_l	Dominated?
s	1	-	0	13	no
1	2	1	0+10=10	13-11=2	no
2	3	1	0-5=-5	13-1=12	no
	6	2	10-5=5	2-1=1	no
3	4	1	0+6=6	13-8=5	no
	8	3	-5+6=1	12-8=4	yes
t	5	1	0+0=0	13-0=13	no
	7	2	10+0=10	2-0=2	no
	9	3	-5+0=-5	12-0=12	yes, by 5
	10	6	5+0=5	1-0=1	yes, by 7 and 11
	11	4	6+0=6	5-0=5	no

(a) From this solution protocol, reconstruct the sequence of labels that leads to the optimal solution and give this solution as well as its objective function value! (2 P.)

(b) Explain why label 6 is not extended to a label sitting at node 3! (2 P.)

(c) Explain why label 8 is dominated and give the number of the label that dominates it! (2 P.)