Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
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Exam on
**Operations Research**
Summer Term 2021

**Step 1: Enter your data below:**

First name:

Last name:

Course of study:

Matriculation number:

Date:

Starting time and ending time of the exam:

**Step 2: Read and sign the following declaration:**

I am aware that attempted cheating will result in the grade "failed" according to the examination regulations. I am enrolled as a regular or exchange student at Leibniz Universität Hannover for the current semester and am not on leave of absence. I expressly declare myself fit to take the examination and agree to the conditions of the examination and the examination procedure. I have taken note of the general instructions. **I confirm that I have completed the online exam independently without the help of others and without helping other exam candidates.** The handwritten text in this file shows my own work.

Place:

Date:

**Signature:**

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**Rating:**

| Task | 1 | 2 | 3 | Bonus points | Sum |
|---|---|---|---|---|---|
| Score | | | | | |

**Hints:**

1. The exam consists of **14** pages (including the front page and this page with hints). Please check that your file is complete. If possible, please write on the hard copy (printed version) of the exam.
2. **If you have to use your own paper, please start each problem on a new page and leave ample space for grading and corrections!**
3. Answer all questions and solve all given problems. For each question or problem, you find next to it in round brackets the number of points that can be earned.
4. You are given 60 minutes to work on the exam and you can score a total of 60 points. You are given additional 30 minutes for downloading, printing, scanning, and uploading your results. The results have to be uploaded by 11:30 pm, May 28, 2021.
5. You may answer the questions using either the German or the English language.
6. **Show your work!** If you use a formula to solve a problem, present it in its general form first!
7. You may use any source, e.g., textbooks, notes taken during the course, files that have been given to you as well as computer programs, but you have to work on your own, not receiving or providing help from or to anybody during the time span of the exam.
8. Name your single PDF file with your solutions which you upload as follows:
   "`OR_SS2021_<MatriculationNumber>_<FirstName>_<LastName>.PDF`"

1. **Elements of modeling (19 P.)**

Assume that a company is providing transportation services. The company is confronted with different transportation jobs which it can accept or refuse. In order to execute the accepted transportation jobs, it has to rent transportation containers that have both volume and weight restrictions. The company has to decide simultaneously which transportation jobs to accept, which containers to rent, and how to assign accepted transportation jobs to rented containers.

Your task is to model the problem formally, using the following assumptions and notation:

- If transportation job $i = 1,...,I$ is accepted, it leads to a revenue $r_i$. Transportation job $i$ has a volume $v_i$ and a weight $w_i$.
- A binary variable $X_i$ assumes a value of 1 if transportation job $i$ is accepted and 0 otherwise.
- If container $j = 1,...,J$ is rented, it leads to a cost $c_j$. The total volume of transportation jobs assigned to container $j$ must not exceed the volume limit $v_j^{\text{lim}}$. In addition, the total weight of transportation jobs assigned to container $j$ must not exceed the weight limit $w_j^{\text{lim}}$.
- A binary variable $U_j$ assumes a value of 1 if container $j$ is rented and 0 otherwise.
- A binary variable $Y_{ij}$ assumes a value of 1 if transportation job $i$ is assigned to container $j$ for transportation and 0 otherwise.

Given these assumptions and using this notation, develop elements of algebraic models as follows:

(a) Assume that the objective is to maximize the profit resulting from the decisions to accept transportation jobs, to rent containers, and to assign accepted jobs to rented containers. State the problem formally (i.e., algebraically) and explain the model verbally. (10 P.)

(b) State formally via linear constraint(s) that if the company accepts transportation job 2, it also has to accept transportation jobs 21 and 32. (2 P.)

(c) State formally via linear constraint(s) that if the company accepts transportation job 8, it must not accept transportation jobs 12 and 15. (2 P.)

(d) State formally via linear constraint(s) that if the company accepts transportation job 5, this transportation job is assigned to either container 2, 7, or 11. (1 P.)

(e) State formally via linear constraint(s) that no container holds both transportation jobs 13 and 14 as they must not be transported together in the same container. (1 P.)

(f) Referring to the nature of this problem, explain which type of algorithm is required to solve instances of this problem to proven optimality! (3 P.)

2. **Linear programming, simplex algorithm, and improving search** **(26 P.)**

Consider the following linear program in standard form:

$$\text{Max } 4x_1 + 6x_2 \tag{1}$$

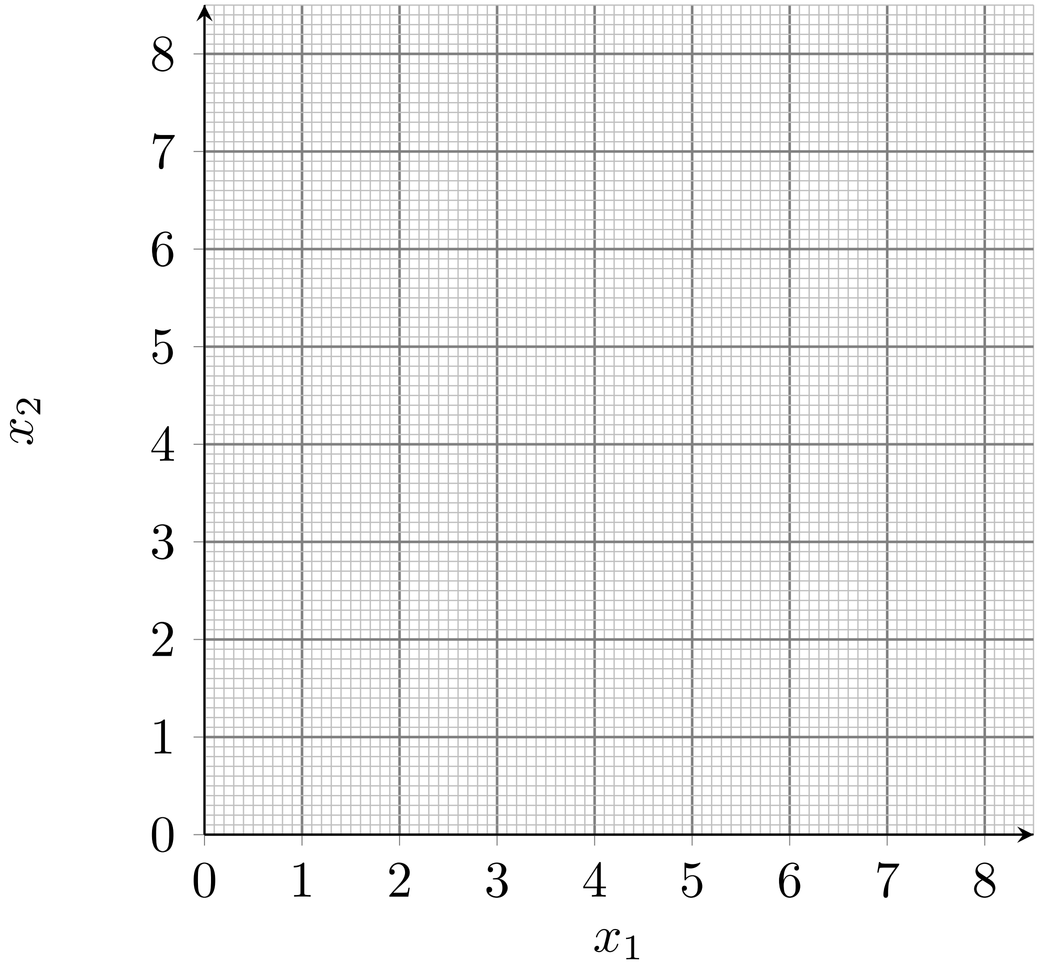
s.t.

$$6x_1 + 4x_2 + x_3 = 24 \tag{2}$$

$$x_1 + 2x_2 + x_4 = 8 \tag{3}$$

$$x_i \geq 0, \qquad i \in \{1,2,3,4\} \tag{4}$$

(a) Assume that the standard form formulation of the linear program originated from a linear program with two "less-than-or-equal-to" constraints after introducing two slack variables $x_3$ and $x_4$. Solve that "original problem" graphically by drawing into the following figure. Give the optimal solution and its objective function value. (5 P.)



(b) Assume that in the current phase of a simplex search, variable $x_1$ and $x_4$ are basic. Compute the complete current solution vector and the objective function value for this current solution vector. (3 P.)

(c) Now compute all corresponding simplex directions leading to a neighboring feasible basic solution. State which variable will leave and which will enter the basis for each of the direction vectors. (6 P.)

(d) Now that you have constructed the collection of simplex directions that can be pursued from the current basis solution without loosing feasibility, determine the change of the objective function value (aka the "reduced cost") associated with nonbasic variable $x_j$ and its direction vector which increases $x_j$. Determine the direction vector which is improving our current solution the most. (3 P.)

(e) Using the chosen improving direction vector, determine the maximum step size $\lambda$, the new solution, and the objective function value for the new solution. (3 P.)

(f) Now state the dual of the linear program (1) - (4), using appropriately defined dual variables $v_1$ and $v_2$. What can you say about the optimal solutions of the primal and the dual problem? (6 P.)

3. **Branch & Bound** **(15 P.)**

Consider the following integer program:

$$\text{Max } 2x_1 + 5x_2 \quad (5)$$

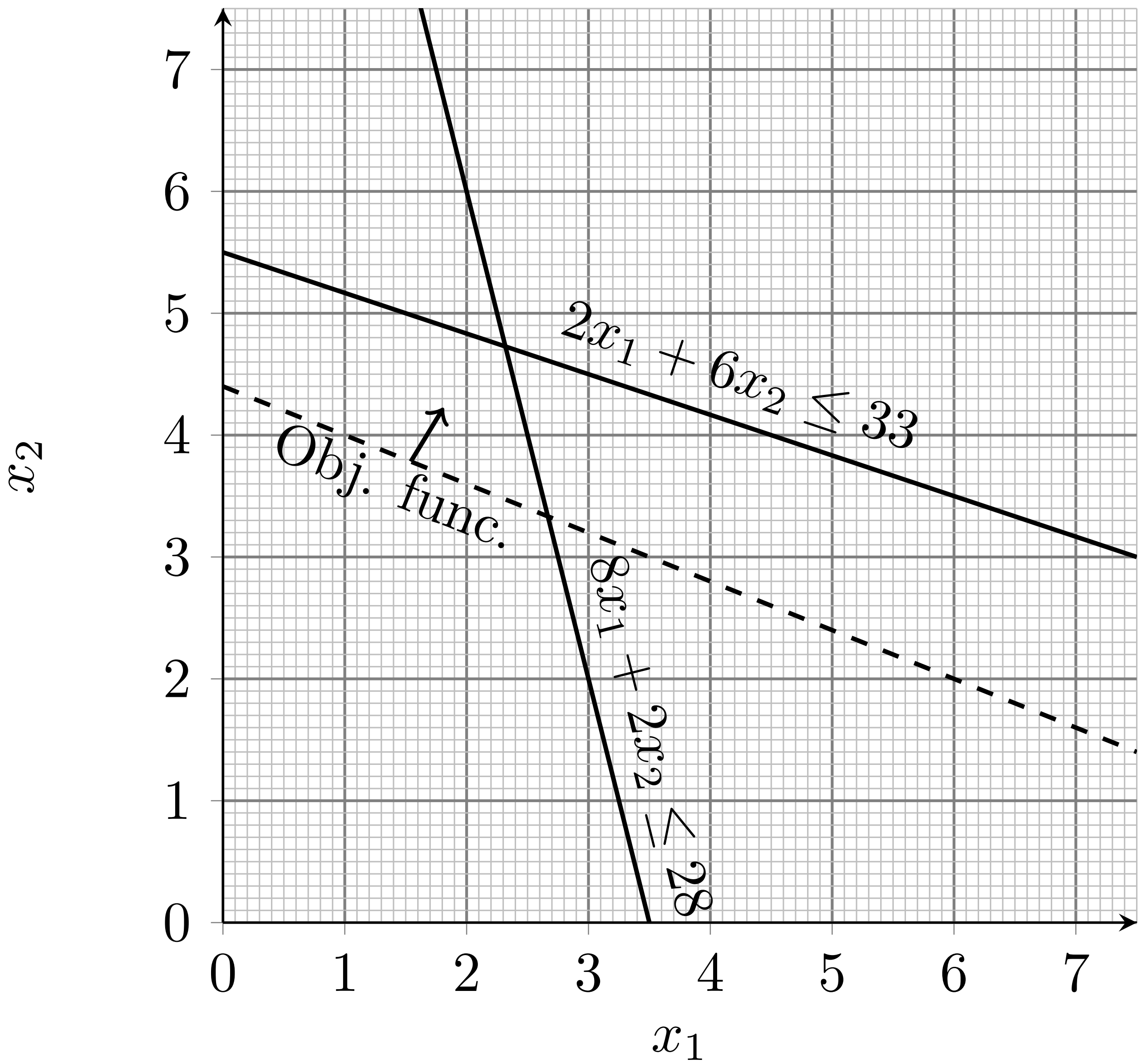
s.t.

$$8x_1 + 2x_2 \leq 28 \quad (6)$$

$$2x_1 + 6x_2 \leq 33 \quad (7)$$

$$x_i \in \{0, 1, 2, 3, 4, \dots\}, \qquad i \in \{1, 2\} \quad (8)$$

The following figure shows the visualization of the objective function and the constraints:



Determine the optimal solution of this integer maximization problem by applying a branch&bound algorithm according to the the following specifications:

- Perform a depth-first search!
- Start with incumbent solution $(x_1 = 0,\ x_2 = 0)$.
- If in the linear programming relaxation of a candidate problem both variables $x_1$ and $x_2$ should be fractional, break ties in favor of the second variable, i.e., branch on $x_2$.
- When branching on a fractional variable $x_i$, create the first new candidate problem by rounding up and the second new problem (to be study late in a depth-first search) by rounding down.
- Number the problems according to the sequence in which you determine their relaxations and make branching or bounding decisions in the decision tree based on the solution of the candidate problem.

Hints and tasks:

- You can determine the values of relaxed variables $x_1$ and $x_2$ in a relaxation of a candidate problem to a sufficient degree of accuracy by modifying and reading from the figure on the previous page. On this basis you can compute to a sufficient degree of accuracy the objective function values as well.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.