Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
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Exam on
Operations Research
Winter Term 2020-2021

Step 1: Enter your data below:

First name:

Last name:

Course of study:

Matriculation number:

Date:

Starting time and ending time of the exam:

Step 2: Read and sign the following declaration:

I am aware that attempted cheating will result in the grade “failed” according to the examination regulations. I am enrolled as a regular or exchange student at Leibniz Universität Hannover for the current semester and am not on leave of absence. I expressly declare myself fit to take the examination and agree to the conditions of the examination and the examination procedure. I have taken note of the general instructions. **I confirm that I have completed the online exam independently without the help of others and without helping other exam candidates.** The handwritten text in this file shows my own work.

Place:

Date:

Signature:

Rating:

Task	1	2	3	Bonus points	Sum
Score					

Hints:

1. The exam consists of **13** pages (including the front page and this page with hints). Please check that your file is complete.
2. Answer all questions and solve all given problems. For each question or problem, you find next to it in round brackets the number of points that can be earned.
3. You are given 60 minutes to work on the exam and you can score a total of 60 points. You are given additional 30 minutes for downloading, printing, scanning, and uploading your results. The results have to be uploaded by 04:00 pm, February 10, 2021.
4. You may answer the questions using either the German or the English language.
5. **Show your work!** If you use a formula to solve a problem, present it in its general form first!
6. You may use any source, e.g., textbooks, notes taken during the course, files that have been given to you as well as computer programs, but you have to work on your own, not receiving or providing help from or to anybody during the time span of the exam.

1. **Elements of modeling (19 P.)**

   Assume that a company is buying, processing, and selling a single product. With the exception of surplus selling on the spot market, both buying and selling decisions are of an **all-or-nothing nature**. To model this problem formally, use the following assumptions and notation:

   - If a customer $i = 1, \ldots, I$ is being served, this customer's entire demand d_i has to be met and a revenue r_i for this customer's entire demand is earned.
   - A binary variable X_i assumes a value of 1 if customer i is being served and 0 otherwise.
   - If a supplier $j = 1, \ldots, J$ is being selected, this supplier's entire supply s_j has to be bought and a cost c_j for this supplier's entire supply is incurred.
   - A binary variable Y_j assumes a value of 1 if supplier j is being used and 0 otherwise.
   - The company cannot sell more than it buys.
   - The surplus quantity $Q \geq 0$ of procured minus sold product units can be sold at a spot market for unit revenue ur.
   - Before being sold, the entire quantity sold to customers $i = 1, \ldots, I$ or at the stock market has to be processed at a facility with limited regular capacity Cap.
   - The regular capacity Cap can be extended by using extra (overtime) capacity O at the processing facility at a unit cost oc.

   Given these assumptions and using this notation, develop elements of algebraic models as follows:

   (a) Assume that the objective is to maximize the profit resulting from the procurement, sales, and overtime decisions subject to the balance constraints on the material and the capacity constraint of the facility. State the problem formally (i.e., algebraically) and explain the model verbally. (10 P.)

(b) State formally via linear constraint(s) that if the company serves customer 3, it also serves customers 4 and 5. (2 P.)

(c) State formally via linear constraint(s) that if the company buys from supplier 7, it does not buy from supplier 1 or 3. (2 P.)

(d) State formally via linear constraint(s) that if the company serves customer 11, it does buy from supplier 8. (1 P.)

(e) State formally via linear constraint(s) that if the company buys from supplier 12, it does not serve customer 23. (1 P.)

(f) Referring to the nature of this problem, explain which type of algorithm is required to solve instances of this problem to proven optimality! (3 P.)

2. **Linear programming, simplex algorithm, and improving search** **(26 P.)**

Consider the following linear program in standard form:

$$\text{Max } 3x_1 + 2x_2 \tag{1}$$

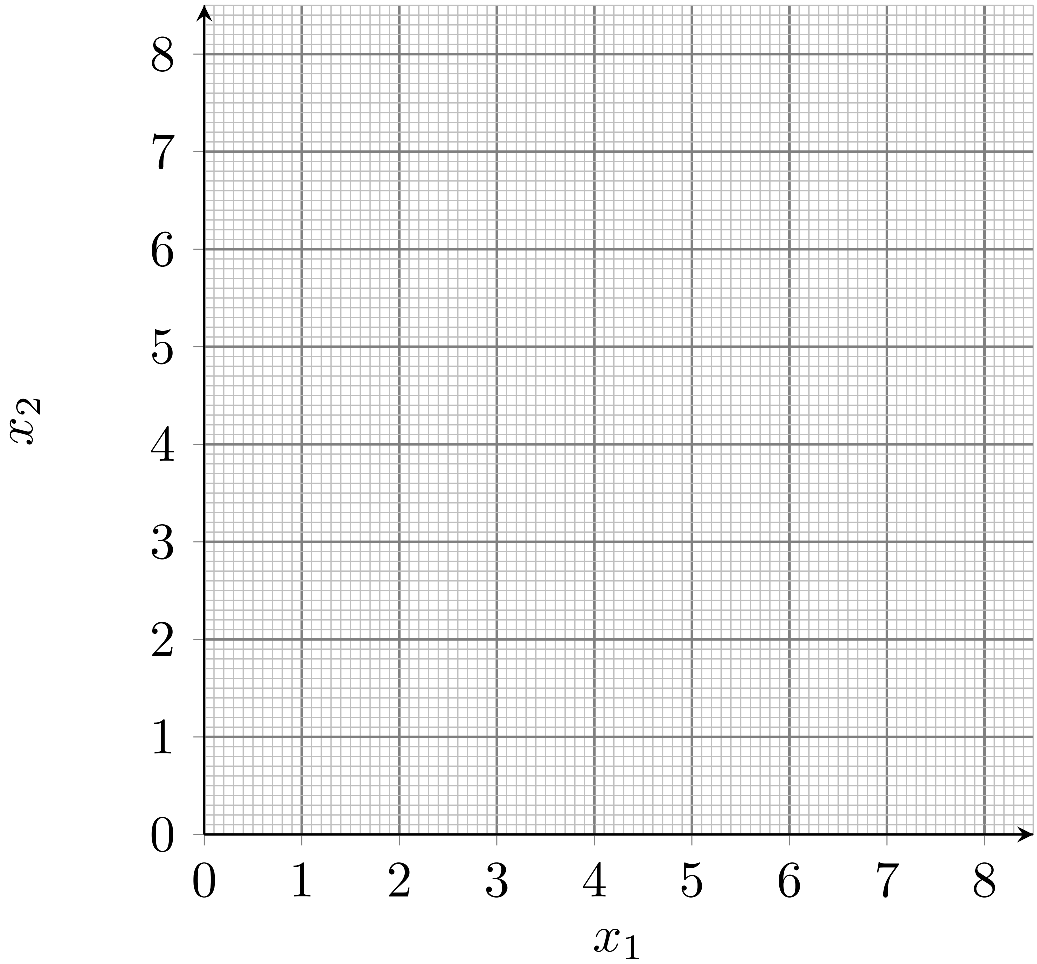
s.t.

$$4x_1 + 6x_2 + x_3 = 24 \tag{2}$$

$$2x_1 + x_2 + x_4 = 8 \tag{3}$$

$$x_i \geq 0, \qquad i \in \{1,2,3,4\} \tag{4}$$

(a) Assume that the standard form formulation of the linear program originated from a linear program with two "less-than-or-equal-to" constraints after introducing two slack variables x_3 and x_4. Solve that "original problem" graphically by drawing into the following figure. Give the optimal solution and its objective function value. (5 P.)



(b) Assume that in the current phase of a simplex search, variable x_2 and x_4 are basic. Compute the complete current solution vector and the objective function value for this current solution vector. (3 P.)

(c) Now compute all corresponding simplex directions leading to a neighboring feasible basic solution. State which variable will leave and which will enter the basis for each of the direction vectors. (6 P.)

(d) Now that you have constructed the collection of simplex directions that can be pursued from the current basis solution without loosing feasibility, determine the change of the objective function value (aka the “reduced cost”) associated with nonbasic variable x_j and its direction vector which increases x_j. Determine the direction vector which is improving our current solution the most. (3 P.)

(e) Using the chosen improving direction vector, determine the maximum step size λ, the new solution, and the objective function value for the new solution. (3 P.)

(f) Now state the dual of the linear program (1) - (4), using appropriately defined dual variables v_1 and v_2. What can you say about the optimal solutions of the primal and the dual problem? (6 P.)

3. **Branch & Bound** **(14 P.)**

Consider the following integer program:

$$\text{Max } 10x_1 + 4x_2 \tag{5}$$

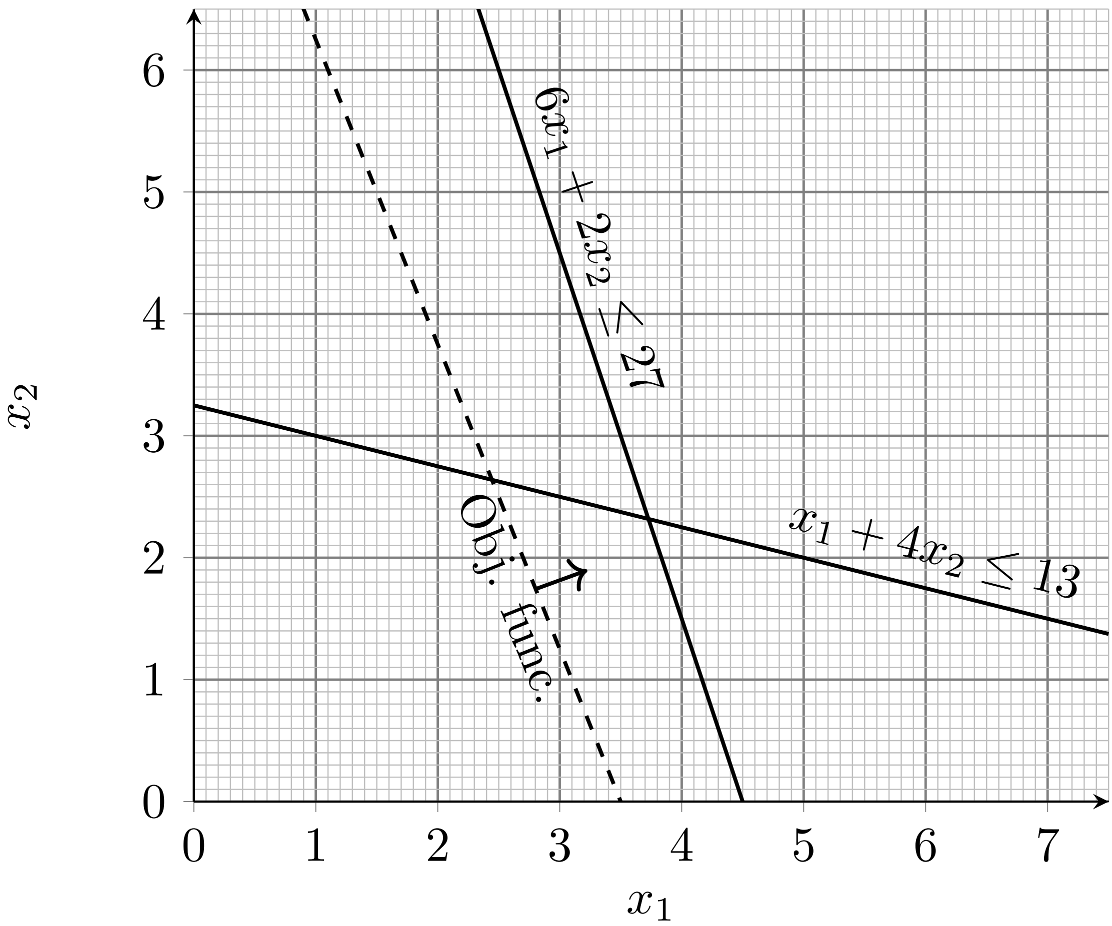
s.t.

$$x_1 + 4x_2 \leq 13 \tag{6}$$

$$6x_1 + 2x_2 \leq 27 \tag{7}$$

$$x_i \in \{0, 1, 2, 3, 4, \ldots\}, \qquad i \in \{1, 2\} \tag{8}$$

The following figure shows the visualization of the objective function and the constraints:



Determine the optimal solution of this integer maximization problem by applying a branch&bound algorithm according to the the following specifications:

- Perform a depth-first search!
- Start with incumbent solution $(x_1 = 0,\ x_2 = 0)$.
- If in the linear programming relaxation of a candidate problem, both variables x_1 and x_2 should be fractional, break ties in favor of the first variable, i.e., branch on x_1.
- When branching on a fractional variable x_i, create the first new candidate problem by rounding up and the second new problem by rounding down.
- Number the problems immediately as you create them in the branching process.

Hints and tasks:

- You can determine the values of relaxed variables x_1 and x_2 in a relaxation of a candidate problem to a sufficient degree of accuracy by modifying and working with the figure on the previous page. On this basis you can compute to a sufficient degree of accuracy the objective function values as well.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- In addition, provide a list stating the sequence in which the candidate problems are considered in the branch&bound process.
- Give the optimal solution and the optimal objective function value.